Comments on “Cartier type”

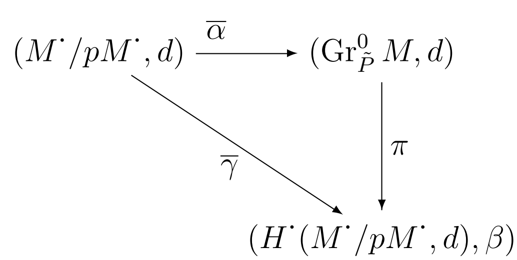
November 29, 2018

L. Illusie has pointed out that it is very unusual for a Dieudonné complex do be both saturated and of Cartier type, a remark he attributes to A. Matthew. Here is another view of this somewhat disturbing comment and a suggested terminological variation that may ameliorate the psychological distress it causes.

Let $(M^\cdot, d, F)$ be a Dieudonné complex. We assume that each term is p-torsion free. Then F induces a morphism of complexes:

$$\alpha\colon (M^\cdot, d) \to \eta_p(M^\cdot, d).$$

Let P denote the p-adic filtration on $M^\cdot$ and let $\tilde{P}$ denote its décalée. Thus $\eta_p(M^\cdot, d) = \tilde{P}^0(M^\cdot, d)$. It is easy to check that $\tilde{P}^1 M^\cdot = p\tilde{P}^0 M^\cdot$, so that $\mathrm{Gr}^0_{\tilde{P}}(M^\cdot, d) \cong \tilde{P}^0 M^\cdot \otimes (\mathbf{Z}/p\mathbf{Z})$. We have a commutative diagram of complexes:



Here $\overline{\alpha}$ identifies with $\alpha \otimes \mathrm{id}_{\mathbf{Z}/p\mathbf{Z}}$, π is the (surjective) quasi-isomorphism defined by Deligne, and β is the Bockstein differential. If $x \in M^i$, then $\alpha(x) = p^i F(x)$; if $\overline{x}$ is the class of x in M^i/pM^i, then $\overline{\gamma}(\overline{x})$ is the class of $F(x)$, and if $z \in \mathrm{Gr}^0_{\tilde{P}} M$ is the class of $p^i y \in \tilde{P}^0 M^i$, then $dy \in pM^{i+1}$ and $\pi(z)$ is the class of the image of y in $H^i(M^\cdot/pM^\cdot, d)$. Thus $\overline{\gamma}$ takes $\overline{x}$ to the class of $F(x)$.

Let us consider the following conditions;

1. The map $\overline{\gamma}$ is an isomorphism of graded abelian groups (hence of complexes), i.e., $(M^{\cdot}, d, F)$ is of Cartier type.

2. The map α is an isomorphism of graded abelian groups (hence of complexes), i.e., $(M^{\cdot}, d, F)$ is saturated.

3. The map $\overline{\gamma}$ is a quasi-isomorphism. Let's then say that $(M^{\cdot}, d, F)$ is of quasi-Cartier type. Since π is always a quasi-isomorphism, we see that $(M^{\cdot}, d, F)$ is of quasi-Cartier type if and only if $\overline{\alpha}$ is a quasi-isomorphism.

4. The map α is a quasi-isomorphism. Let us then say that $(M^{\cdot}, d, F)$ is quasi-saturated.

If M is of Cartier type, it is of quasi-Cartier type. If M is saturated, it is quasi-saturated. If the terms of M are p-adically complete, then α is a quasi-isomorphism if and only if $\overline{\alpha}$ is. Thus M is quasi-saturated if and only if it is of quasi-Cartier type. If M is quasi-saturated, the map $M \to Sat(M)$ is a quasi-isomorphism.

Proposition 1 *Suppose that $(M^{\cdot}, d, F)$ is saturated and of Cartier type and that each term of $M^{\cdot}$ is p-adically separated. Then F is surjective and $d = 0$.*

Proof: If α is an isomorphism, then $\overline{\alpha}$ is also an isomorphism, and hence if also $(M^{\cdot}, d, F)$ is of Cartier type, π is an isomorphism. Say $x \in M^i$. Then $p^{i+1}x \in \tilde{P}^0 M^i$ and the class of $p^{i+1}x$ in $\mathrm{Gr}^0_{\tilde{P}} M^i$ lies in $\mathrm{Ker}(\pi)$. Since π is an isomorphism, it follows that $p^{i+1}x \in \tilde{P}^1 M^i = p\tilde{P}^0 M^i$, and hence $p^i x \in \tilde{P}^0 M^i$. Thus $\tilde{P}^0 M^i = p^i M^i$. Since α is surjective, it follows that $F\colon M^i \to M^i$ is surjective. Since dF is divisible by p and M^{i+1} is p-adically separated, it follows that $d = 0$. □